SUMMATION FORMULAE FOR q-WATSON TYPE ${}_4\phi_3$-SERIES

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Abstract. According to the method of series rearrangement, we establish two families of summation formulae for q-Watson type ${}_4\phi_3$-series.

1. Introduction

For two complex numbers x and q, define the q-shifted factorial by

$$(x;q)_0=1 \quad \text{and} \quad (x;q)_n=\prod_{i=0}^{n-1}(1-xq^i) \quad \text{when} \quad n\in\mathbb{N}.$$

Its fraction form reads as

$$\begin{bmatrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix} q \Big]_n=\frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}.$$

Following Gasper and Rahman [4], the basic hypergeometric series can be defined by

$${}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\Big|\, q;z\right]=\sum_{k=0}^{\infty}\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_r \end{matrix}\Big|\, q\right]_k z^k$$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then the q-Watson formula due to Andrews [1] and the q-Watson formula due to Jain [5, Eq. 3.17] can be stated, respectively, as follows:

$${}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, c \end{matrix}\Big|\, q;q\right]=\begin{cases} c^s\left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix}\Big|\, q^2\right]_s, & n=2s; \\ 0, & n=1+2s, \end{cases} \tag{1}$$

$${}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{-2n} \end{matrix}\Big|\, q;q\right]=\left[\begin{matrix} qa, qc \\ q, qac \end{matrix}\Big|\, q^2\right]_n. \tag{2}$$

Inspired by the method due to Chu [3], we shall establish the following two families of summation formulae for q-Watson type ${}_4\phi_3$-series:

$${}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix}\Big|\, q;q\right] \quad \text{and} \quad {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q^{\varepsilon}\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix}\Big|\, q;q\right],$$

2010 Mathematics Subject Classification: Primary 05A19 and Secondary 33D15.

Key words and phrases. Basic hypergeometric series; The terminating ${}_6\phi_5$-series identity; Summation formulae for q-Watson type ${}_4\phi_3$-series.

$$ {}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n} \end{matrix} \Big| q; q\right] \quad \text{and} \quad {}_4\phi_3\left[\begin{matrix} a, c, q^{\varepsilon-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n} \end{matrix} \Big| q; q\right] $$

where the disturbing parameter ε is a nonnegative integer throughout the paper. Eight concrete summation formulae corresponding $1 \le \varepsilon \le 2$ will be exhibited.

2. Summation formulae for q-Andrews-Watson type ${}_4\phi_3$-series

2.1. Letting $a \to c/q$, $b \to q^{-k}$ and $c \to \infty$ for the terminating ${}_6\phi_5$-series identity (cf. Gasper and Rahman [4, p. 42]):

$$ {}_6\phi_5\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ q^{-\varepsilon} \\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ q^{1+\varepsilon}a \end{matrix} \Big| q; \frac{q^{1+\varepsilon}a}{bc}\right] = \left[\begin{matrix} qa, qa/bc \\ qa/b, qa/c \end{matrix} \Big| q\right]_{\varepsilon}, \tag{3} $$

we obtain the following equation:

$$ \sum_{i=0}^{\varepsilon} \begin{bmatrix} k \\ i \end{bmatrix} q^{(i+\varepsilon-1)i} c^i \left[\begin{matrix} cq^{k+i} \\ cq^i \end{matrix} \Big| q\right]_{\varepsilon-i} \frac{1-cq^{2i-1}}{1-cq^{i-1}} \left[\begin{matrix} q^{-\varepsilon} \\ q^{\varepsilon}c \end{matrix} \Big| q\right]_i = 1. $$

Then we can proceed as follows:

$$ \begin{aligned} {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| q; q\right] &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}-nk} \left[\begin{matrix} q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| q\right]_k \\ = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}-nk} & \left[\begin{matrix} q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| q\right]_k \\ \times \sum_{i=0}^{\varepsilon} \begin{bmatrix} k \\ i \end{bmatrix} q^{(i+\varepsilon-1)i} c^i & \left[\begin{matrix} cq^{k+i} \\ cq^i \end{matrix} \Big| q\right]_{\varepsilon-i} \frac{1-cq^{2i-1}}{1-cq^{i-1}} \left[\begin{matrix} q^{-\varepsilon} \\ q^{\varepsilon}c \end{matrix} \Big| q\right]_i. \end{aligned} $$

Interchanging the summation order, we can manipulate the last double sum as

$$ \begin{aligned} {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| q; q\right] &= \sum_{i=0}^{\varepsilon} \begin{bmatrix} n \\ i \end{bmatrix} q^{(i+\varepsilon-1)i} c^i \frac{1-cq^{2i-1}}{1-cq^{i-1}} \left[\begin{matrix} q^{-\varepsilon} \\ q^{\varepsilon}c \end{matrix} \Big| q\right]_i \\ &\times \sum_{k=i}^{n} (-1)^k \begin{bmatrix} n-i \\ k-i \end{bmatrix} q^{\binom{k+1}{2}-nk} \left[\begin{matrix} q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{i}c \end{matrix} \Big| q\right]_k. \end{aligned} $$

Shifting the summation index $k \to i + j$ for the sum on the last line, we get the relation:

$$ \begin{aligned} {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| q; q\right] &= \sum_{i=0}^{\varepsilon} q^{(i+\varepsilon)i} c^i \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c, q^{i-1}c \end{matrix} \Big| q\right]_i \\ &\times {}_4\phi_3\left[\begin{matrix} q^{i-n}, q^{1+n+i}a, q^i\sqrt{c}, -q^i\sqrt{c} \\ q^{1+i}\sqrt{a}, -q^{1+i}\sqrt{a}, q^{2i}c \end{matrix} \Big| q; q\right]. \end{aligned} \tag{4} $$

Evaluating the ${}_4\phi_3$-series on the last line by (1), we establish the following summation theorem.

Theorem 1. *For two complex numbers $\{a, c\}$ and a nonnegative integer ε, there holds the summation formula for q-Andrews-Watson type ${}_4\phi_3$-series:*

$$ \begin{aligned} {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| q; q\right] &= \sum_{i=0}^{\varepsilon} q^{(\varepsilon+n)i} c^{\frac{n+i}{2}} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c, q^{i-1}c \end{matrix} \Big| q\right]_i \\ &\times \left[\begin{matrix} q, q^2a/c \\ q^{2+2i}a, q^{1+2i}c \end{matrix} \Big| q^2\right]_{\frac{n-i}{2}} \chi(n-i \equiv_2 0) \end{aligned} $$

where $n - i \equiv_2 0$ stands for the congruence relation $n - i = 0$ (mod 2) and χ is the logical function defined by $\chi(true) = 1$ and $\chi(false) = 0$ otherwise.

Corollary 2 ($\varepsilon = 1$ in Theorem 1)**.**

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, qc \end{matrix} \Big| \, q; q\right] = \begin{cases} c^s \left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix} \Big| \, q^2\right]_s, & n = 2s; \\ c^{1+s}\left[\begin{matrix} q \\ qc \end{matrix} \Big| \, q^2\right]_{1+s}\left[\begin{matrix} q^2a/c \\ q^2a \end{matrix} \Big| \, q^2\right]_s, & n = 1+2s. \end{cases} $$

Corollary 3 ($\varepsilon = 2$ in Theorem 1)**.**

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^2c \end{matrix} \Big| \, q; q\right] $$

$$ = \begin{cases} \left\{1 + \frac{q^2c(1-c)(1-q^{2s})(1-q^{1+2s}a)}{(1-q^2c)(1-q^{1+2s}c)(1-q^{2s}a/c)}\right\} c^s \left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix} \Big| \, q^2\right]_s, & n = 2s; \\ \frac{1-q^2}{1-q^2c}\, c^{1+s}\left[\begin{matrix} q^3, q^2a/c \\ q^2a, q^3c \end{matrix} \Big| \, q^2\right]_s, & n = 1+2s. \end{cases} $$

2.2. Letting $a \to c/q$, $b \to q^{-k}$ and $c \to \sqrt{c}$ for (3), we attain the following equation:

$$ \sum_{i=0}^{\varepsilon}(-1)^i q^{\varepsilon+\binom{i}{2}} c^{\frac{i-\varepsilon}{2}} \frac{1-cq^{2i-1}}{1-c\,q^{i+\varepsilon-1}} \left[\begin{matrix} q^{-\varepsilon} \\ q \end{matrix} \Big| \, q\right]_i \left[\begin{matrix} q^{1-\varepsilon}/\sqrt{c} \\ q^{2-\varepsilon-i}/c \end{matrix} \Big| \, q\right]_\varepsilon $$
$$ \times \frac{< q^k; q >_i < cq^{k+\varepsilon-1}; q >_{\varepsilon-i}}{(q^k\sqrt{c}; q)_\varepsilon} = 1 $$

where we have used the symbol:

$$ < x; q >_0 = 1 \quad \text{and} \quad < x; q >_n = \prod_{i=0}^{n-1}(1 - xq^{-i}) \quad \text{when} \quad n \in \mathbb{N}. $$

Then we can proceed as follows:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q^\varepsilon\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c \end{matrix} \Big| \, q; q\right] = \sum_{k=0}^{n}\left[\begin{matrix} q^{-n}, q^{1+n}a, q^\varepsilon\sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c \end{matrix} \Big| \, q; q\right]_k q^k $$
$$ = \sum_{k=0}^{n}\left[\begin{matrix} q^{-n}, q^{1+n}a, q^\varepsilon\sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c \end{matrix} \Big| \, q; q\right]_k q^k \sum_{i=0}^{\varepsilon}(-1)^i q^{\varepsilon+\binom{i}{2}} c^{\frac{i-\varepsilon}{2}} \frac{1-cq^{2i-1}}{1-cq^{i+\varepsilon-1}} $$
$$ \times \left[\begin{matrix} q^{-\varepsilon} \\ q \end{matrix} \Big| \, q\right]_i \left[\begin{matrix} q^{1-\varepsilon}/\sqrt{c} \\ q^{2-\varepsilon-i}/c \end{matrix} \Big| \, q\right]_\varepsilon \frac{< q^k; q >_i < cq^{k+\varepsilon-1}; q >_{\varepsilon-i}}{(q^k\sqrt{c}; q)_\varepsilon}. $$

Interchanging the summation order, we can reformulate the last double sum as

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q^\varepsilon\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c \end{matrix} \Big| \, q; q\right] = \sum_{i=0}^{\varepsilon}(-1)^i q^{\varepsilon+\binom{i}{2}} c^{\frac{i-\varepsilon}{2}} \frac{1-cq^{2i-1}}{1-cq^{i+\varepsilon-1}} $$
$$ \times \left[\begin{matrix} q^{-\varepsilon} \\ q \end{matrix} \Big| \, q\right]_i \left[\begin{matrix} q^{1-\varepsilon}/\sqrt{c} \\ q^{2-\varepsilon-i}/c \end{matrix} \Big| \, q\right]_\varepsilon $$
$$ \times \sum_{k=i}^{n}\left[\begin{matrix} q^{-n}, q^{1+n}a, q^\varepsilon\sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c \end{matrix} \Big| \, q; q\right]_k q^k \frac{< q^k; q >_i < cq^{k+\varepsilon-1}; q >_{\varepsilon-i}}{(q^k\sqrt{c}; q)_\varepsilon}. $$

Shifting the summation index $k \to i + j$ for the sum on the last line, we achieve the relation:

$$ {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q^\varepsilon\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c \end{matrix} \Big| \, q; q\right] $$
$$ = \sum_{i=0}^{\varepsilon}(-1)^i q^{\varepsilon i+\binom{i+1}{2}} c^{\frac{i}{2}} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^\varepsilon c, q^{i-1}c \end{matrix} \Big| \, q\right]_i $$
$$ \times\ {}_4\phi_3\left[\begin{matrix} q^{i-n}, q^{1+n+i}a, q^i\sqrt{c}, -q^i\sqrt{c} \\ q^{1+i}\sqrt{a}, -q^{1+i}\sqrt{a}, q^{2i}c \end{matrix} \Big| \, q; q\right]. \tag{5} $$

Evaluating the ${}_4\phi_3$-series on the last line by (1), we found the following summation theorem.

Theorem 4. *For two complex numbers $\{a, c\}$ and a nonnegative integer ε, there holds the summation formula for q-Andrews-Watson type ${}_4\phi_3$-series:*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q^{\varepsilon}\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c \end{matrix} \Big| \, q; q\right] \\
&= \sum_{i=0}^{\varepsilon}(-1)^i q^{(\varepsilon+n)i-\binom{i}{2}} c^{\frac{n}{2}} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q, q\sqrt{a}, -q\sqrt{a}, q^{\varepsilon}c, q^{i-1}c \end{matrix} \Big| \, q\right]_i \\
&\times \left[\begin{matrix} q, q^2a/c \\ q^{2+2i}a, q^{1+2i}c \end{matrix} \Big| \, q^2\right]_{\frac{n-i}{2}} \chi(n-i \equiv_2 0).
\end{aligned}
$$

Corollary 5 ($\varepsilon = 1$ in Theorem 4)**.**

$$
{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, qc \end{matrix} \Big| \, q; q\right] = \begin{cases} c^s \left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix} \Big| \, q^2\right]_s, & n = 2s; \\ -c^{\frac{1}{2}+s}\left[\begin{matrix} q \\ qc \end{matrix} \Big| \, q^2\right]_{1+s} \left[\begin{matrix} q^2a/c \\ q^2a \end{matrix} \Big| \, q^2\right]_s, & n = 1+2s. \end{cases}
$$

Corollary 6 ($\varepsilon = 2$ in Theorem 4)**.**

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+n}a, q^2\sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, q^2c \end{matrix} \Big| \, q; q\right] \\
&= \begin{cases} \left\{1 + \frac{q(1-c)(1-q^{2s})(1-q^{1+2s}a)}{(1-q^2c)(1-q^{1+2s}c)(1-q^{2s}a/c)}\right\} c^s \left[\begin{matrix} q, q^2a/c \\ q^2a, qc \end{matrix} \Big| \, q^2\right]_s, & n = 2s; \\ \frac{q^2-1}{1-q^2c}\, c^{\frac{1}{2}+s}\left[\begin{matrix} q^3, q^2a/c \\ q^2a, q^3c \end{matrix} \Big| \, q^2\right]_s, & n = 1+2s. \end{cases}
\end{aligned}
$$

3. Summation formulae for q-Jain-Watson type ${}_4\phi_3$-series

3.1. Performing the replacement $a \to aq^{-1-n}$ for (4), we have

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix} q^{-n}, a, \sqrt{c}, -\sqrt{c} \\ \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{\varepsilon}c \end{matrix} \Big| \, q; q\right] &= \sum_{i=0}^{\varepsilon} q^{(i+\varepsilon)i}c^i \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, a, \sqrt{c}, -\sqrt{c} \\ q, \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{\varepsilon}c, q^{i-1}c \end{matrix} \Big| \, q\right]_i \\
&\times {}_4\phi_3\left[\begin{matrix} q^{i-n}, q^ia, q^i\sqrt{c}, -q^i\sqrt{c} \\ q^i\sqrt{q^{1-n}a}, -q^i\sqrt{q^{1-n}a}, q^{2i}c \end{matrix} \Big| \, q; q\right].
\end{aligned}
$$

Employing the substitutions $c \to q^{-2n}$ and $q^{-n} \to c$ for the last equation, we obtain the following relation:

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n} \end{matrix} \Big| \, q; q\right] &= \sum_{i=0}^{\varepsilon} q^{(i+\varepsilon-2n)i} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, -q^{-n}, a, c \\ q, \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n}, q^{i-1-2n} \end{matrix} \Big| \, q\right]_i \\
&\times {}_4\phi_3\left[\begin{matrix} q^ia, q^ic, q^{i-n}, -q^{i-n} \\ q^i\sqrt{qac}, -q^i\sqrt{qac}, q^{2i-2n} \end{matrix} \Big| \, q; q\right].
\end{aligned}
$$

Evaluating the ${}_4\phi_3$-series on the last line by (2), we establish the following summation theorem.

Theorem 7. *For two complex numbers $\{a, c\}$ and a nonnegative integer ε with $0 \le \varepsilon \le n$, there holds the summation formula for q-Jain-Watson type ${}_4\phi_3$-series:*

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n} \end{matrix} \Big| \, q; q\right] &= \sum_{i=0}^{\varepsilon} q^{(i+\varepsilon-2n)i} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, -q^{-n}, a, c \\ q, \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n}, q^{i-1-2n} \end{matrix} \Big| \, q\right]_i \\
&\times \left[\begin{matrix} q^{1+i}a, q^{1+i}c \\ q, q^{1+2i}ac \end{matrix} \Big| \, q^2\right]_{n-i}.
\end{aligned}
$$

Corollary 8 ($\varepsilon = 1$ in Theorem 7: $n \geq 1$)**.**

$$
{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{1-2n} \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n + \left[\begin{matrix} a, c \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n .
$$

Corollary 9 ($\varepsilon = 2$ in Theorem 7: $n \geq 2$)**.**

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{2-2n} \end{matrix} \,\Big|\, q; q\right] &= \frac{(1+q)(1-q^{1-2n})}{1-q^{2-2n}} \left[\begin{matrix} a, c \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n \\
&+ \Big\{1 + \tfrac{q(1-a)(1-c)(1-q^{-2n})}{(1-q^{2-2n})(1-aq^{2n-1})(1-cq^{2n-1})}\Big\} \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n .
\end{aligned}
$$

3.2. Performing the replacement $a \to aq^{-1-n}$ for (5), we have

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} q^{-n}, a, q^{\varepsilon}\sqrt{c}, -\sqrt{c} \\ \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{\varepsilon}c \end{matrix} \,\Big|\, q; q\right] \\
&= \sum_{i=0}^{\varepsilon} (-1)^i q^{\varepsilon i + \binom{i+1}{2}} c^{\frac{i}{2}} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, a, \sqrt{c}, -\sqrt{c} \\ q, \sqrt{q^{1-n}a}, -\sqrt{q^{1-n}a}, q^{\varepsilon}c, q^{i-1}c \end{matrix} \,\Big|\, q\right]_i \\
&\times\ {}_4\phi_3\left[\begin{matrix} q^{i-n}, q^i a, q^i\sqrt{c}, -q^i\sqrt{c} \\ q^i\sqrt{q^{1-n}a}, -q^i\sqrt{q^{1-n}a}, q^{2i}c \end{matrix} \,\Big|\, q; q\right].
\end{aligned}
$$

Employing the substitutions $c \to q^{-2n}$ and $q^{-n} \to c$ for the last equation, we get the following relation:

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{\varepsilon-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n} \end{matrix} \,\Big|\, q; q\right] \\
&= \sum_{i=0}^{\varepsilon} (-1)^i q^{(\varepsilon-n)i + \binom{i+1}{2}} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, -q^{-n}, a, c \\ q, \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n}, q^{i-1-2n} \end{matrix} \,\Big|\, q\right]_i \\
&\times\ {}_4\phi_3\left[\begin{matrix} q^i a, q^i c, q^{i-n}, -q^{i-n} \\ q^i\sqrt{qac}, -q^i\sqrt{qac}, q^{2i-2n} \end{matrix} \,\Big|\, q; q\right].
\end{aligned}
$$

Evaluating the ${}_4\phi_3$-series on the last line by (2), we found the following summation theorem.

Theorem 10. *For two complex numbers $\{a, c\}$ and a nonnegative integer ε with $0 \leq \varepsilon \leq n$, there holds the summation formula for q-Jain-Watson type ${}_4\phi_3$-series:*

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} a, c, q^{\varepsilon-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n} \end{matrix} \,\Big|\, q; q\right] \\
&= \sum_{i=0}^{\varepsilon} (-1)^i q^{(\varepsilon-n)i + \binom{i+1}{2}} \left[\begin{matrix} q^{-\varepsilon}, q^{-n}, -q^{-n}, a, c \\ q, \sqrt{qac}, -\sqrt{qac}, q^{\varepsilon-2n}, q^{i-1-2n} \end{matrix} \,\Big|\, q\right]_i \\
&\times \left[\begin{matrix} q^{1+i}a, q^{1+i}c \\ q, q^{1+2i}ac \end{matrix} \,\Big|\, q^2\right]_{n-i} .
\end{aligned}
$$

Corollary 11 ($\varepsilon = 1$ in Theorem 10: $n \geq 1$)**.**

$$
{}_4\phi_3\left[\begin{matrix} a, c, q^{1-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{1-2n} \end{matrix} \,\Big|\, q; q\right] = \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n - q^n \left[\begin{matrix} a, c \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n .
$$

Corollary 12 ($\varepsilon = 2$ in Theorem 10: $n \geq 2$)**.**

$$
\begin{aligned}
{}_4\phi_3\left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ \sqrt{qac}, -\sqrt{qac}, q^{2-2n} \end{matrix} \,\Big|\, q; q\right] &= \frac{(1+q)(1-q^{2n-1})}{q^{n-1}-q^{1-n}} \left[\begin{matrix} a, c \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n \\
&+ \Big\{1 - \tfrac{(1-a)(1-c)(1-q^{2n})}{(1-q^{2-2n})(1-aq^{2n-1})(1-cq^{2n-1})}\Big\} \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \,\Big|\, q^2\right]_n .
\end{aligned}
$$

With the change of ε, more concrete formulae could be derived from Theorems 1, 4, 7 and 10. Considering that the resulting identities will become complicated, we shall not display them one by one. Altough Chu [3] gave more formulae for hypergeometric series, we end our paper in the present form because of the restriction of basic hypergeometric series.

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